Overfullness of edge-critical graphs with small minimal core degree

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Abstract

Let G be a simple graph. Let $\Delta(G)$ and $\chi'(G)$ be the maximum degree and the chromatic index of G, respectively. We call G *overfull* if $|E(G)|/\lfloor |V(G)|/2\rfloor > \Delta(G)$, and *critical* if $\chi'(H) < \chi'(G)$ for every proper subgraph H of G. Clearly, if G is overfull then $\chi'(G) = \Delta(G)+1$. The *core* of G, denoted by G_Δ, is the subgraph of G induced by all its maximum degree vertices. We believe that utilizing the core degree condition could be considered as an approach to attack the overfull conjecture. Along

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this direction, we in this paper show that for any integer $k \geq 2$, if G is critical with $\Delta(G) \geq \frac{2}{3}n + \frac{3k}{2}$ and $\delta(G_\Delta) \leq k$, then G is overfull.

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1 Introduction

We will mainly follow the notation from [9]. Graphs in this paper are simple, i.e., finite, undirected, without loops or multiple edges. Let G be a graph and let $[k] = \{i \mid 1 \leq i \leq k \text{ and } i \in \mathbb{Z}\}$ for a nonnegative integer k. A *k-edge-coloring* of G is a mapping φ: $E(G) \to [k]$ that assigns to every edge e of G a color $\varphi(e) \in [k]$ such that no two adjacent edges receive the same color. Denote by $\mathcal{C}^k(G)$ the set of all k-edge-colorings of G. The *chromatic index* $\chi'(G)$ is the least integer $k \geq 0$ such that $\mathcal{C}^k(G) \neq \emptyset$. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of G, respectively. In 1960's, Vizing [12] and, independently, Gupta [6] proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. This leads to a natural classification of graphs. Following Fiorini and Wilson [4], we say a graph G is of *class 1* if $\chi'(G) = \Delta(G)$ and of *class 2* if $\chi'(G) = \Delta(G) + 1$. Holyer [8] showed that it is NP-complete to determine whether an arbitrary graph is of class 1. The problem of deciding which graphs are of class one, and which are of class two, is known as the Classification Problem [4, 9].

A graph G is *critical* if $\chi'(H) < \chi'(G)$ for every proper subgraph H of G. In investigating the Classification Problem, critical graphs are of particular interest. A critical class 2 graph is called *Δ-critical* if $\Delta(G) = \Delta$. An edge $e \in E(G)$ is called a *critical* edge if $\chi'(G - e) < \chi'(G)$. Clearly, if G is critical, then every edge of G is a critical edge. For convenience, we denote $|V(G)|$ by n throughout this paper. Since every matching of G has at most $\lfloor n/2 \rfloor$ edges, $\chi'(G) \geq |E(G)|/\lfloor n/2 \rfloor$. A graph G is *overfull* if $|E(G)|/\lfloor n/2 \rfloor > \Delta(G)$. In 1986, Chetwynd and Hilton [3] conjectured that if G is a class 2 critical graph with $\Delta(G) > \frac{n}{3}$, then G is overfull. This conjecture is known as the Overfull Conjecture.

The *core* of a graph G, denoted by G_Δ, is the subgraph induced by all its maximum degree vertices. Vizing [12] proved that if G_Δ has at most two vertices then G is class 1. Fournier [5] generalized Vizing's result by showing that if G_Δ is acyclic then G is class 1. Thus a necessary condition for a graph to be class 2 is to have a core that contains cycles. A long-standing conjecture of Hilton and Zhao [7] claims that for a connected class 2 graph G with $\Delta \geq 4$, if $\Delta(G_\Delta) \leq 2$, then G is overfull. This conjecture was recently confirmed by the authors [1]. Another paper of Cao, Chen, and Shan [2] extended the result above by changing the maximum core degree condition to a minimum core degree condition, and showed that for any critical class 2 graph G, if $\delta(G_\Delta) \leq 2$ and $\Delta(G) > n/2 + 1$, then G is overfull.

Along this direction, we prove the following result and verify the overfull conjecture for critical graphs with a more general minimum core degree condition, and we hope to use similar ideas to attack the overfull conjecture in the future. For example, if we can improve the coefficient of k in Theorem 1.1 from $3/2$ to $1/12$, then the overfull conjecture holds for all graphs with maximum degree $\Delta \geq 3n/4$.

Theorem 1.1. *Let $k \geq 2$ be a positive integer and G be a Δ-critical graph of order n. If $\Delta \geq \frac{2}{3}n + \frac{3k}{2}$ and $\delta(G_\Delta) \leq k$, then G is overfull.*

2 Preliminaries

This section is divided into two subsections. In the first subsection we introduce some basic notation and terminologies. In the second subsection we introduce the traditional Vizing fan and generalize it to a larger structure.

2.1 Basic notation and terminologies

Let G be a graph with maximum degree Δ, let $e \in E(G)$ be a critical edge, and let $\varphi \in \mathcal{C}^\Delta(G-e)$. For a vertex $v \in V(G)$, define the two color sets

$$\varphi(v) = \{\varphi(f) \,:\, f \neq e \text{ is incident to } v\} \quad \text{ and } \quad \overline{\varphi}(v) = [\Delta] \setminus \varphi(v).$$

We call $\varphi(v)$ the set of colors *present* at v and $\overline{\varphi}(v)$ the set of colors *missing* at v. If $|\overline{\varphi}(v)| = 1$, we will also use $\overline{\varphi}(v)$ to denote the color missing at v. Let $N(v)$ be the collection of all the neighbors of v, $N_{<\Delta}(v)$ be the collection of neighbors of v with degree less than Δ, and $N_\Delta(v)$ be the collection of neighbors of v with degree exactly Δ.

For a vertex set $X \subseteq V(G)$, define $\overline{\varphi}(X) = \bigcup_{v \in X} \overline{\varphi}(v)$ to be the set of missing colors of X. The set X is called *elementary* w.r.t. φ or simply called *elementary* if $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for every two distinct vertices $u, v \in X$. In the rest of this paper, we may not always mention the coloring φ if it is clearly understood.

For a color α, the edge set $E_\alpha = \{e \in E(G) \,|\, \varphi(e) = \alpha\}$ is called a *color class*. Clearly, E_α is a *matching* of G (possibly empty). For two distinct colors α, β, the subgraph of G induced by $E_\alpha \cup E_\beta$ is a union of disjoint paths and even cycles, which are referred to as (α,β)-*chains* of G w.r.t. φ. For a vertex v, let $C_v(\alpha,\beta,\varphi)$ denote the unique (α,β)-chain containing v. If $C_v(\alpha,\beta,\varphi)$ is a path, we just write it as $P_v(\alpha,\beta,\varphi)$. The latter is commonly used when we know that $|\overline{\varphi}(v) \cap \{\alpha,\beta\}| = 1$. If we interchange the colors α and β on an (α,β)-chain C of G, we briefly say that the new coloring is obtained from φ by an (α,β)-*swap* on C, and we write it as φ/C. This operation is called a *Kempe change*. If $\alpha \in \overline{\varphi}(v)$, by doing operation $\alpha \to \beta$ at v we mean the Kempe change $\varphi/P_v(\alpha,\beta,\varphi)$. Note that $P_v(\alpha,\beta,\varphi)$ could be empty when $\alpha,\beta \in \overline{\varphi}(v)$ and $\alpha \to \beta$ at v does nothing in this case. We say two vertices x and y are (α,β)-*linked* if they belong to the same (α,β)-chain. Moreover, when $x = y$, for convenience we still say x and y are (α,β)-linked even if $\alpha,\beta \in \overline{\varphi}(x)$.

2.2 Linear Sequence, Shifting, and Extended Vizing fan

The fan argument was introduced by Vizing [10, 11] in his proof of the classic results on the upper bounds for chromatic indices. Let G be a class 2 graph with maximum degree Δ, $e = rs$ be a critical edge of G, and let $\varphi \in \mathcal{C}^{\Delta}(G-e)$. For an integer $p \geq 0$, a sequence $F = (r, e_0, s_0, e_1, s_1, \dots, e_p, s_p)$ alternating between distinct vertices and edges is called a *Vizing fan* at r with respect to e and φ if $s_0 = s$, $e_0 = e$ and for each $i \in [p]$, the edge $e_i = rs_i$ satisfies $\varphi(e_i) \in \overline{\varphi}(s_h)$ for some $0 \leq h \leq i-1$. For the purpose of generalization in this paper, we include the vertex r in F comparing to the definition of a Vizing fan in the book [9]. Let q be a nonnegative integer. A *linear sequence* at r from s_0 to s_q in G, denoted by $L = (r, e_0, s_0, e_1, s_1, \dots, e_q, s_q)$, is a sequence of distinct vertices and edges such that $\varphi(e_i) \in \overline{\varphi}(s_{i-1})$ for $i \in [q]$. Denote by $V(L)$ and $E(L)$ respectively the set of vertices and edges contained in L. A *shifting* from s_i to s_j in the linear sequence $L = (r, e_0, s_0, e_1, s_1, \dots, e_q, s_q)$ is an operation that replaces the current color of e_t by the color of e_{t+1} for each $i \leq t \leq j-1$ with $0 \leq i < j \leq q$. Note that shifting from s_i to s_j does not change the color of e_j where $e_j = rs_j$, so the resulting coloring will not be a proper coloring. In our proof we will treat e_j separately to avoid this problem. The following result regarding a Vizing fan can be found in [9, Theorem 2.1].

Lemma 2.1. *Let G be a class 2 graph, $e = rs_0$ be a critical edge and $\varphi \in \mathcal{C}^{\Delta}(G-e)$. If F is a Vizing fan w.r.t. e and φ, then $V(F)$ is elementary.*

Note that e_0 may not be e in a linear sequence, but a linear sequence with $e_0 = e$ is also a Vizing fan at r. Moreover, for any $s_i \in V(F)$ with $i \in [p]$, the Vizing fan $F = (r, e_0, s_0, e_1, s_1, \dots, e_p, s_p)$ contains a linear sequence at r from s_0 to s_i. A linear sequence at r with $\varphi(e_0) = \tau$ is called a *τ-sequence*. In our proof we will add some linear sequences not contained in a Vizing fan to enlarge it. We say a Vizing fan F at r is *maximal* w.r.t. e and φ if there is no Vizing fan at r w.r.t. e and φ containing F as a proper subsequence. We say a Vizing fan F at r is *maximum* w.r.t. e if $|V(F)|$ is maximum among all Vizing fans at r w.r.t. e over all colorings $\varphi \in \mathcal{C}^{\Delta}(G-e)$. Clearly if F is maximum at r w.r.t. e, it is also maximal w.r.t. e and the coloring φ where F is obtained. Let F be a maximal Vizing fan at r w.r.t. e and φ. A τ-sequence L at r is said to be *outside* of F if $V(L) \cap V(F) = \{r\}$. For an integer $t \geq 0$, we say a τ-sequence $L = (r, f_0, v_0, f_1, v_1, \dots, f_t, v_t)$ at r outside of F is *extremal* if v_t is the only vertex v_j with index $0 \leq j \leq t$ such that either $\overline{\varphi}(v_j) \cap (\cup_{i=0}^{j-1} \overline{\varphi}(v_i) \cup \overline{\varphi}(V(F)) \cup \{\tau\}) \neq \emptyset$ or $\overline{\varphi}(v_j) = \emptyset$. Since a τ-sequence cannot be enlarged forever, it must be a subsequence of some extremal τ-sequence. Moreover, exactly one of the followings must happen for an extremal τ-sequence L:

(a) $V(L) \cup V(F)$ is elementary and $\{\tau\} = \overline{\varphi}(v_t)$. In this case we say L is of *Type A*.

(b) $\overline{\varphi}(v_t) \cap \overline{\varphi}(V(F)) \neq \emptyset$. In this case we say L is of *Type B*.

(c) $\overline{\varphi}(v_i) \cap \overline{\varphi}(V(F)) = \emptyset$ for all $0 \leq i \leq t$, and $V(L)$ is not elementary. In this case there exists a color $\alpha \in (\overline{\varphi}(v_i) \cap \overline{\varphi}(v_j)) - \overline{\varphi}(V(F))$ for some $0 \leq i \leq j \leq t$ and we say L is of *Type C*.

(d) $\overline{\varphi}(v_t) = \emptyset$ and $V(L) \cup V(F)$ is elementary. In this case $d(v_t) = \Delta$ and we say L is of *Type D*.

See the following figure 1 for examples of 4 types of extremal τ-sequences, where a dash line represents a color missing at a vertex.

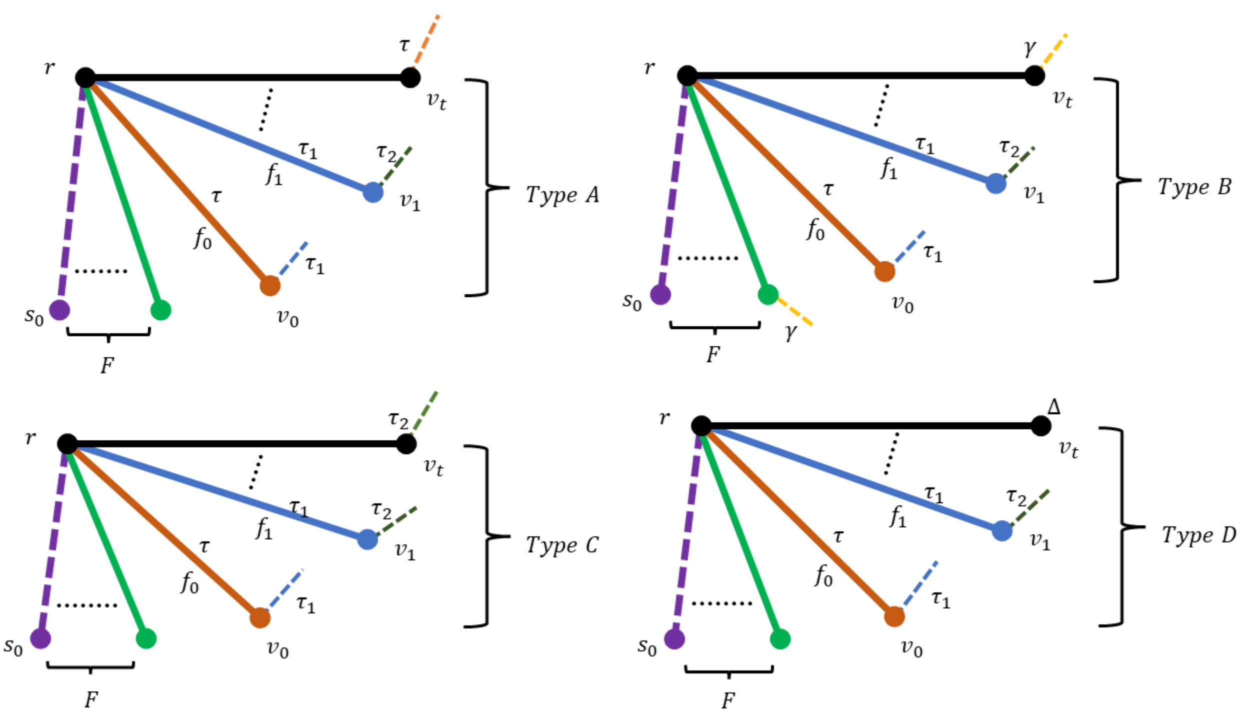

Figure 1: Examples of τ-sequences

From now on we will not mention "at r" when we refer to a Vizing fan or a linear sequence if it creates no confusion. Additionally, when we refer to a linear sequence outside of F, we always mean an extremal one unless specified otherwise.

Let G be a class 2 graph, $e = rs_0$ be a critical edge and $\varphi \in \mathcal{C}^{\Delta}(G-e)$. Let $F = (r, e_0, s_0, e_1, s_1, \dots, e_p, s_p)$ be a Vizing fan centered at r under the coloring φ. Clearly a linear sequence L at r from s_0 to s_q with $q \in [p]$ defines a linear order $\preceq_L$ on vertices in L. By $s_a \prec_L s_b$, we mean $s_a \preceq_L s_b$ and $s_a \neq s_b$. Since $V(F)$ is elementary by Lemma 2.1, it's easy to see that all the linear sequences at r starting from s_0 to s_q for some $q \in [p]$ together induce a partial order $\preceq_F$ by $\alpha \preceq_F \alpha$ for the every color $\alpha \in \overline{\varphi}(V(F))$, and $\alpha \preceq_F \beta$ for two different colors $\alpha, \beta \in \overline{\varphi}(V(F))$ if there exists a linear sequence L at r starting from s_0 to some s_q with $q \in [p]$ such that an edge $e' \in E(L)$ with $\varphi(e') = \alpha$ comes before a vertex $v \in V(L)$ with $\beta \in \overline{\varphi}(v)$ along L. Moreover, for any color $\alpha \in \overline{\varphi}(V(F))$, there is a unique vertex $v \in V(F)$ such that $\alpha \in \overline{\varphi}(v)$ since $V(F)$ is elementary. Let $v_F(\alpha)$ denote such vertex v. We have the following lemma as a direct consequence of Lemma 2.1.

Lemma 2.2. *Let G be a class 2 graph, $e = rs_0$ be a critical edge and $\varphi \in \mathcal{C}^{\Delta}(G-e)$. Let F be a maximal Vizing fan at r w.r.t. e and φ. Then for any two colors $\alpha, \beta \in \overline{\varphi}(V(F))$, we have the following statements:*

(a) If $v_F(\alpha) = r$, then $v_F(\alpha)$ and $v_F(\beta)$ are (α, β)-linked.

(b) If α and β are incomparable along $\preceq_F$, then $v_F(\alpha)$ and $v_F(\beta)$ are (α, β)-linked.

(c) If $\alpha \preceq_F \beta$ and $v_F(\alpha)$ and $v_F(\beta)$ are not (α, β)-linked, then $P_{v_F(\beta)}(\alpha, \beta, \varphi)$ must contain the vertex r.

(d) If $v \in V(F)$ and $v \neq r$, then F contains at least $|\overline{\varphi}(v)|$ many Δ-degree neighbors of r.

Proof. To prove (a) we assume $v_F(\alpha) = r$. Note that if $v_F(\beta) = r$, we are done by definition. So we may assume that $v_F(\beta) \neq r$. If $v_F(\alpha)$ and $v_F(\beta)$ are not (α, β)-linked, then by $\beta \to \alpha$ at $v_F(\beta)$, we have a non-elementary Vizing fan F' from r to $v_F(\beta)$ contradicting Lemma 2.1. Thus (a) holds.

For (b) we assume that α and β are incomparable along $\preceq_F$. Note that there are two linear sequences L_1 and L_2 at r from s_0 to $v_F(\alpha)$ and $v_F(\beta)$, respectively. Since α and β are incomparable along $\preceq_F$, and $v_F(\alpha)$ and $v_F(\beta)$ are the last vertices for L_1 and L_2 respectively, L_1 and L_2 do not contain any edge colored by α or β. Now by $\beta \to \alpha$ at $v_F(\beta)$, we have a new coloring and we denote the new coloring by φ_1. Since no edge in L_1 and L_2 is colored by either α or β under φ, L_1 and L_2 are still linear sequences under φ_1. Let F' be a maximal Vizing fan w.r.t. e and φ_1. Then L_1 and L_2 are all contained in F', giving a non-elementary Vizing fan contradicting Lemma 2.1.

If (c) fails, since $\alpha \preceq_F \beta$, we can just do $\beta \to \alpha$ at $v_F(\beta)$ to get a non-elementary Vizing fan contradicting Lemma 2.1.

To see (d), we assume $\alpha \in \overline{\varphi}(v)$ with $v \in V(F)$ and $v \neq r$. Since $V(F)$ is elementary by Lemma 2.1 and F is maximal, every color α in $\overline{\varphi}(v)$ induces at least one maximal α-sequence L_α ending with a unique Δ-degree vertex in F, giving at least $|\overline{\varphi}(v)|$ many Δ-degree neighbors of r in F. □

The following Vizing's Adjacency Lemma is a direct consequence of Lemmma 2.2(d).

Lemma 2.3 (Vizing's Adjacency Lemma(VAL))**.** *Let G be a class 2 graph with maximum degree Δ. If $e = xy$ is a critical edge of G, then x is adjacent to at least $\Delta - d(y) + 1$ Δ-vertices from $V(G) \setminus \{y\}$.*

Let G be a class 2 graph, $e = rs_0$ be a critical edge and $F = (r, e_0, s_0, e_1, s_1, \ldots, e_p, s_p)$ be a maximum Vizing fan at r w.r.t. e, and let $\varphi \in \mathcal{C}^\Delta(G - e)$ be the coloring where F is obtained. We call a color β a *stopping* color at r if r has a Δ-degree neighbor x with $\varphi(rx) = \beta$. Let K be the set of all stopping colors at r. Since G is class 2, e is critical, and F is maximum and elementary, F must contain some Δ-degree neighbors of r. So there exists a vertex $s_h \in V(F)$ and stopping color β such that $\beta \in \overline{\varphi}(s_h)$. We let $K_F = K - \overline{\varphi}(V(F))$ and call colors in K_F *stopping colors outside* of F. By a slightly abuse of notation, in this paper, a *union* of two sequences A and B, denoted by $A \cup B$, is the sequence obtained by joining the sequence B to A after the last element of A. We now fix a vertex $s_h \in V(F)$ with a stopping color $\beta \in \overline{\varphi}(s_h)$. Let F' be the union of all the $\varphi(rv)$-sequences outside of F, where v is any vertex in the set $N_{<\Delta}(s_h) \cap N(r)$ with $\varphi(vs_h) \notin K_F$. Then we call the sequence $F \cup F'$ an *extended Vizing fan* w.r.t. F and s_h. See figure 2 for an extended Vizing fan with F' being a single type A τ-sequence with $\varphi(vs_h) = \pi$, where a dash line represents a color missing at a vertex. For simplification of notation, we did not indicate s_h in the notation F', even though F' relies on a fixed vertex s_h. The following Lemma 2.4 is a key lemma in our proof and it is a natural generalization of Lemmas 2.1 and 2.2 on $F \cup F'$. It is worth pointing out that Lemma 2.4 can be easily generalized further along this direction if we allow F' to be the union of all the $\varphi(rv)$-sequences outside of F such that $v \in N_{<\Delta}(s_h) \cap N(r)$ with $\varphi(vs_h) \notin K_F$ for every vertex s_h having any stopping color $\beta \in \overline{\varphi}(s_h)$.

Lemma 2.4. *Let G be a class 2 graph, $e = rs_0$ be a critical edge and $F = (r, e_0, s_0, e_1, s_1, \ldots, e_p, s_p)$ be a maximum Vizing fan at r w.r.t. e, and let $\varphi \in \mathcal{C}^\Delta(G - e)$ be the coloring where F is obtained. Let $F \cup F'$ be an extended Vizing fan w.r.t. F and $s_h \in V(F)$, where β is a stopping color with $\beta \in \overline{\varphi}(s_h)$. Then the following holds.*

(a) $V(F \cup F')$ is elementary.

(b) For two colors $1 \in \overline{\varphi}(r)$ and $\gamma \in \overline{\varphi}(F \cup F') - K_F$, the vertices r and $v_{F\cup F'}(\gamma)$ are $(1, \gamma)$-linked, where $v_{F\cup F'}(\gamma)$ is the unique vertex in $V(F \cup F')$ with $\gamma \in \overline{\varphi}(v_{F\cup F'}(\gamma))$.

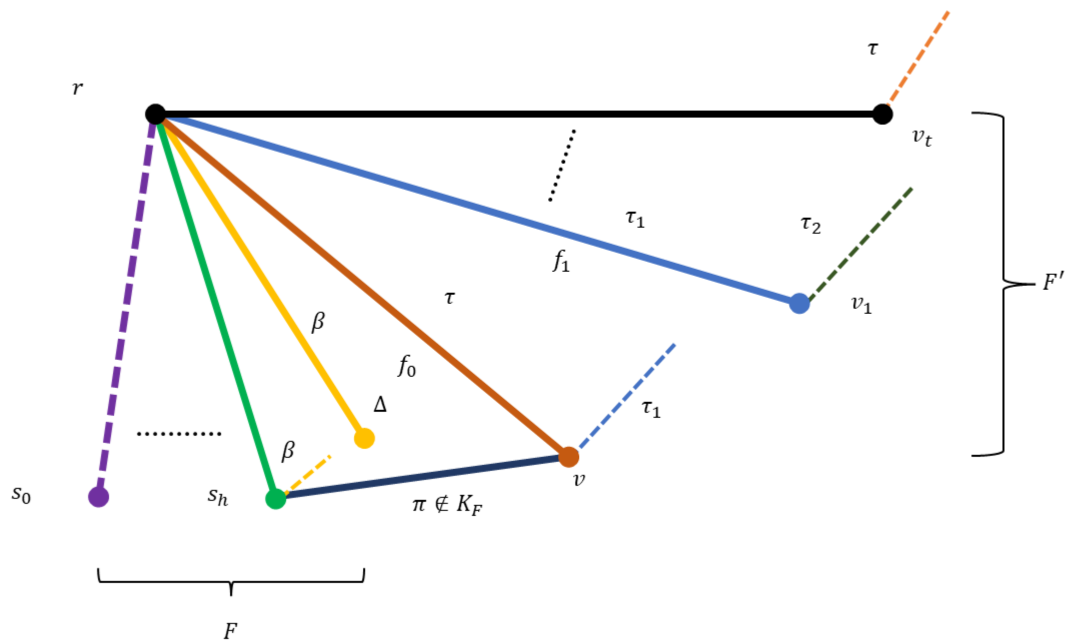

Figure 2: An extended Vizing fan with F' being a single type A τ-sequence with $\varphi(vs_h) = \pi$

(c) For each color $\gamma \in \overline{\varphi}(V(F'))$, the vertices s_h and $v_{F'}(\gamma)$ are (β, γ)-linked, where $v_{F'}(\gamma)$ is the unique vertex in $V(F')$ with $\gamma \in \overline{\varphi}(v_{F'}(\gamma))$.

(d) Let γ be a color in $\overline{\varphi}(V(F')) \cap K_F$ and let $L = (r, rv_0, v_0, ..., v_t)$ be a $\varphi(rv_0)$-sequence in F' such that $v_{F'}(\gamma) \in V(L)$. If $\varphi(s_h v_0) \neq 1$, then r is $(1, \gamma)$-linked to $v_{F'}(\gamma)$. If $\varphi(s_h v_0) = 1$, then $v_{F'}(\gamma)$ is (ζ, γ)-linked to $v_F(\zeta)$ for any color $\zeta \in \overline{\varphi}(V(F)) \cap K$.

The proof of Lemma 2.4 will be given in Section 4. We call a vertex r *light* if $d(r) = \Delta$ and $d_{G_\Delta}(r) = \delta(G_\Delta)$. The next lemma is the main tool used in the proof of Theorem 1.1.

Lemma 2.5. *Let G be a critical class 2 graph with $\delta(G_\Delta) = k$, $|V(G)| = n$, and $\Delta \geq \frac{2}{3}n + \frac{3k}{2}$, let r be a light vertex, and let $e = rs$ be a critical edge with $d(s) \leq \Delta - 1$. Let $\varphi \in \mathcal{C}^\Delta(G - e)$ be a coloring under which there is a maximum Vizing fan centered at r. Then all vertices of degree at least $\Delta - k + 1$ form an elementary set under φ.*

3 Proof of Theorem 1.1

Proof. Let G be a Δ-critical graph of order n and $k \geq 2$ be a positive integer. Furthermore, we assume $\Delta \geq \frac{2}{3}n + \frac{3k}{2}$ with $\delta(G_\Delta) \leq k$. Since $\Delta \geq \frac{2}{3}n + \frac{3k}{2} \geq \frac{2}{3}n + \frac{3\delta(G_\Delta)}{2}$, we will just take $\delta(G_\Delta) = k$ in this proof. Let r be a light vertex of G and s be a neighbor of r with $d(s) \leq \Delta - 1$. Then the edge rs is a critical edge. Let φ be a Δ-edge-coloring of $G - rs$ and F be a maximum Vizing fan centered at r. We first claim that if $V(G)$ is elementary under φ, then G is overfull. Indeed, if G is elementary, then each color can only be missing at most once for vertices in $V(G)$. Since r has at least one missing color, n must be odd as any color missing at r induces a perfect matching of $G - r$. Therefore, each color must be missing exactly once in G as n is odd. Thus G has exactly $(\frac{n-1}{2})\Delta + 1$ many edges since we have Δ many color classes and the edge rs is uncolored. So G is overfull as we claimed.

Now we shall show that $V(G)$ is elementary to confirm that G is overfull in the reminder of this section. By Lemma 2.5, all vertices with degree at least $\Delta - k + 1$ form an elementary set, so we are done if there's no vertex of degree less than $\Delta - k + 1$. Thus we assume otherwise that there is a vertex x with $d(x) \leq \Delta - k$. Since $|N_\Delta(r)| = k$, all the vertices in $N(r)$ have degree at least $\Delta - k + 1$ by applying Lemma 2.3(VAL) to the edge xr. Since $d(x) \leq \Delta - k$, we have $x \notin N(r)$.

We claim that $d(x) \geq \frac{n}{3} + 2k$. Since every edge in G is critical, x is adjacent to at least one maximum degree vertex in G by Lemma 2.3(VAL). Let u be a maximum degree vertex with $ux \in E(G)$. Then $u \neq r$ as $x \notin N(r)$. Since $d(u) = \Delta$, we have $|N(u) \cap N(r)| \geq d(u) + d(r) - |N(u) \cup N(r)| \geq \Delta + \Delta - n \geq \frac{4n}{3} + 3k - n \geq \frac{n}{3} + 3k$. Since $|N_\Delta(r)| = k$, we have $|N_{<\Delta}(u)| \geq \frac{n}{3} + 2k$, and therefore $|N_\Delta(u)| \leq \Delta - \frac{n}{3} - 2k$. Since ux is a critical edge, we have $|N_\Delta(u)| \geq \Delta - d(x) + 1$. So $d(x) \geq \frac{n}{3} + 2k + 1 \geq \frac{n}{3} + 2k$ as claimed.

Since $N_\Delta(r) = k$, we have $d(v) \geq \Delta - k + 1$ for each vertex $v \in N_{<\Delta}(r)$ by by Lemma 2.3(VAL). Recall that by Lemma 2.5, all vertices with degree at least $\Delta - k + 1$ form an elementary set. As $s \in N_{<\Delta}(r)$, we have $|\overline{\varphi}(N_{<\Delta}(r))| \geq |N_{<\Delta}(r)| + 1 \geq \Delta - k + 1$. Since $d(x) \geq \frac{n}{3} + 2k$, we have $|N(r) \cap N(x)| \geq \Delta + \frac{n}{3} + 2k - n \geq \frac{7k}{2}$. Because $|N_\Delta(r)| = k$, it follows that $|N_{<\Delta}(r) \cap N_{<\Delta}(x)| \geq \frac{5k}{2}$. Since $|\overline{\varphi}(N_{<\Delta}(r))| \geq \Delta - k + 1$ and all edges connecting x to vertices in $N_{<\Delta}(r) \cap N_{<\Delta}(x)$ are colored differently, there is a vertex $v \in N_{<\Delta}(r) \cap N_{<\Delta}(x)$ such that $\varphi(vx) = \beta \in \overline{\varphi}(w)$ where $w \in N_{<\Delta}(r)$. Since $d(x) \leq \Delta - k$, $|\overline{\varphi}(x)| \geq k$. Since $|\overline{\varphi}(N_{<\Delta}(r))| \geq \Delta - k + 1$, $|\overline{\varphi}(x) \cap \overline{\varphi}(N_{<\Delta}(r))| \geq 1$. Thus, there exists $\alpha \in \overline{\varphi}(x) \cap \overline{\varphi}(u')$ where $u' \in N_{<\Delta}(r)$. So $d(u') \geq \Delta - k + 1$. Let $1 \in \overline{\varphi}(r)$. We claim that u' is $(1, \alpha)$-linked to r. Otherwise, we have $u' \notin V(F)$ by Lemma 2.2(a). Thus F stays as a maximum Vizing fan after $1 \to \alpha$ at u'. However, we have a contradiction to Lemma 2.5 as now $1 \in \overline{\varphi}(r) \cap \overline{\varphi}(u')$ and $d(u') \geq \Delta - k + 1$. Thus we have as claimed. Then x is not $(1, \alpha)$-linked to r, as $x \notin V(F)$ ($x \notin N(r)$). Thus after doing $\alpha \to 1$ at x, F stays as a maximum Vizing fan. Now we let $\gamma \in \overline{\varphi}(v)$. Similarly as earlier, we see that v is $(1, \gamma)$-linked to r, as otherwise we can do $\gamma \to 1$ at v and reach a contradiction with Lemma 2.5. Hence x is not $(1,\gamma)$-linked to r, as $x \notin V(F)$ ($x \notin N(r)$). Thus we do $1 \to \gamma$ at x. Now $\gamma \in \overline{\varphi}(x) \cap \overline{\varphi}(v)$ and we recolor the edge vx by γ. Note that F stays as a maximum Vizing fan after these two operations. As a result, we have $\beta \in \overline{\varphi}(v) \cap \overline{\varphi}(w)$, a contradiction to Lemma 2.5. Therefore, G has no vertex of degree less than $\Delta - k + 1$ and $V(G)$ is elementary by Lemma 2.5, as desired.

□

4 Proof of Lemma 2.4

Proof. Let G be a class 2 graph, $e = rs_0$ be a critical edge and $F = (r, e_0, s_0, e_1, s_1, \dots, e_p, s_p)$ be a maximum Vizing fan at r w.r.t. e, and let $\varphi \in \mathcal{C}^\Delta(G - e)$ be the coloring where F is obtained. Let $F \cup F'$ be an extended Vizing fan as defined earlier using the vertex s_h with $0 \le h \le p$. Let $1 \in \overline{\varphi}(r)$ and $\beta \in \overline{\varphi}(s_h) \cap K$, where K is the set of stopping colors.

We first prove (a). Assume otherwise that there exist $\alpha \in \overline{\varphi}(x_1) \cap \overline{\varphi}(x_2)$ with $x_1, x_2 \in V(F \cup F')$. We first assume that $\alpha \notin K_F$. Since only one of x_1, x_2 is $(1,\alpha)$-linked to r, so we assume that x_1 is not $(1,\alpha)$-linked to r. By Lemma 2.1, $x_1 \notin V(F)$. By the definition of F', x_1 must be a vertex along a τ-sequence L such that $\varphi(rv) = \tau$ for a vertex v in $N_{<\Delta}(r)$ and $\varphi(vs_h) \notin K_F$, where K_F is the set of stopping colors outside of F. In this case, we do $\alpha \to 1$ at x_1. Note that $\varphi(vs_h)$ may be changed when we did $\alpha \to 1$ at x_1, but we still have $\varphi(vs_h) \notin K_F$ because $1, \alpha \notin K_F$. Moreover, if L contains an edge colored by α and a vertex $x_3 \prec_L x_1$ with $\alpha \in \overline{\varphi}(x_3)$ such that x_1 and x_3 are $(1,\alpha)$-linked, x_1 may no longer belong to the corresponding τ-sequence L after the color switching at x_1. Nonetheless, we can still shift L from v to x_3 and recolor rx_3 by 1 if the earlier mentioned x_3 exists, and we shift L from v to x_1 and recolor rx_1 by 1 if otherwise. Now $\tau \in \overline{\varphi}(v) \cap \overline{\varphi}(r)$ and F is still a Vizing fan. Recall that $\beta \in \overline{\varphi}(s_h) \cap K$. Let $\gamma = \varphi(vs_h)$. So $\gamma \notin K_F$. By Lemma 2.2(a), s_h and r are (τ,β)-linked. Then we do $\tau \to \beta$ at v. As a result, $\beta \in \overline{\varphi}(s_h) \cap \overline{\varphi}(v)$ and the edge vs_h is still colored by γ. If $\gamma \notin \overline{\varphi}(F)$, then we do $\beta \to \gamma$ at s_h. Recall that $\beta \in K_F$ gives us a Δ-degree neighbor of r, say w. Now since $\gamma \notin K_F$, under the new coloring a γ-sequence of at least two vertices can be added to F with removing w from F, resulting in a larger Vizing fan, which is a contradiction to F being maximum w.r.t. e. Thus we may assume $\gamma \in \overline{\varphi}(s_i)$ for a vertex $s_i \in V(F)$. Now we recolor the edge vs_h by β, and γ becomes a missing color at s_h. Since $\beta \in K$, we then have a non-elementary Vizing fan containing both s_h and s_i, a contradiction to Lemma 2.1.

Now we assume that $\alpha \in K_F$. Recall that $\beta \in \overline{\varphi}(s_h) \cap K$. Similarly as earlier, we assume that x_1 is not (α,β)-linked to s_h, and x_1 is added to F' through a τ-sequence at r starting from rv. We do $\alpha \to \beta$ at x_1. Since x_1 is not (α,β)-linked to s_h and $\alpha, \beta \in K$, this process does not change the colors on vs_h and rv, and x_1 still belongs to a τ-sequence at r starting from rv. Since s_h is $(1,\beta)$-linked to r, we can do $\beta \to 1$ at x_1 and this process still keeps the colors of vs_h and rv, and x_1 still belongs to a τ-sequence at r starting from rv. We then have $1 \in \overline{\varphi}(x_1) \cap \overline{\varphi}(r)$ and returned to the previous case of $\alpha \notin K_F$ with 1 in place of α.

To see (b), we just do $\gamma \to 1$ at $v_{F\cup F'}(\gamma)$ if (b) fails. Since $v_{F\cup F'}(\gamma)$ and r are not $(1,\gamma)$-linked, this Kempe change does not involve any edge of $F \cup F'$. Although this Kempe change may change the color on some edge vs_h where a $\varphi(rv)$-sequence is contained in F' by the definition of F', since $1, \gamma \notin K_F$, we still have that the color on the edge rv is not contained in K_F, and as a result, $v_{F\cup F'}(\gamma)$ now belongs to a non-elementary extended Vizing fan (it may be different from $F \cup F'$, but it still contains the vertex $v_{F\cup F'}(\gamma)$), giving a contradiction to (a).

The proof of (c) is similar to the proof of (b), as we can do $\gamma \to \beta$ at $v_{F'}(\gamma)$ if (c) fails. Since $\beta \in K$, $\beta \in \overline{\varphi}(s_h)$ and $v_{F'}(\gamma)$ is not (β,γ)-linked to s_h, $v_{F'}(\gamma)$ now belongs to a non-elementary extended Vizing fan (again it may be different from $F \cup F'$), a contradiction to (a).

If the first part of (d) fails, then by $\gamma \to 1$ at $v_{F'}(\gamma)$, we have a non-elementary extended Vizing fan containing the vertex $v_{F'}(\gamma)$, a contradiction to (a). If the second part of (d) fails, then similarly we do $\gamma \to \zeta$ at $v_{F'}(\gamma)$ and get a non-elementary extended Vizing fan containing the vertex $v_{F'}(\gamma)$. Note that in both parts after the operation on $v_{F'}(\gamma)$, the extended Vizing fan may be different from $F \cup F'$, but it will still contain the vertex $v_{F'}(\gamma)$.

□

5 Proof of Lemma 2.5

Proof. Let G, k, $s = s_0$, r, and the maximum Vizing fan $F = (r, e_0, s_0, e_1, s_1, \ldots, e_p, s_p)$ be as defined in Lemma 2.5 under the coloring $\varphi \in \mathcal{C}^\Delta(G - e_0)$. Recall that K is the set of stopping colors at r and $K_F = K - \overline{\varphi}(V(F))$ is the set of stopping colors outside of F. Then $|K| = k$ as r has core degree k. We denote $|K_F|$ by k'. Let $1 \in \overline{\varphi}(r)$. To prove Lemma 2.5, we assume otherwise that there are two vertices x, x' with degree at least $\Delta - k + 1$ such that $\alpha \in \overline{\varphi}(x) \cap \overline{\varphi}(x')$. Since r is $(1,\alpha)$-linked to exactly one vertex, we may assume that x is not $(1,\alpha)$-linked to r. By Lemma 2.1, $x \notin V(F)$. Thus we do $\alpha \to 1$ at x. As a result, $1 \in \overline{\varphi}(x)$. Assume $\beta \in K \cap \overline{\varphi}(s_h)$ for some h with $0 \le h \le p$. Let $F \cup F'$ be an extended Vizing fan defined with s_h where F' is a collection of all the τ-sequences outside of F such that $\tau = \varphi(rv)$ and $v \in N_{<\Delta}(s_h) \cap N(r)$ with $\varphi(vs_h) \notin K_F$. Since $|N_\Delta(r)| = k$ and rs is critical, all neighbors of r have degree at least $\Delta - k + 1$ by Theorem 2.3(VAL). Thus all vertices in $V(F \cup F')$ have degree at least $\Delta - k + 1$. We have the following claim.

Claim 1. $|N_{<\Delta}(x) \cap V(F \cup F')| \geq k+1$.

By Lemma 2.2(d), F contains at least $|\overline{\varphi}(s_h)|$ many Δ-neighbors of r. Thus we have $k' \leq k - |\overline{\varphi}(s_h)|$. Since $|\overline{\varphi}(s_h)| = \Delta - d_G(s_h)$ when $h > 0$ and $|\overline{\varphi}(s_h)| = \Delta - d_G(s_h) + 1$ when $h = 0$, we have $d_G(s_h) \geq \Delta - |\overline{\varphi}(s_h)|$, and therefore $d_G(s_h) \geq \Delta - (k - k') = \Delta - k + k'$. So $|N(r) \cap N(s_h)| \geq d_G(r) + d_G(s_h) - |V(G)| = \Delta + \Delta - k + k' - n \geq n/3 + 2k + k'$. Since there are at most k' neighbors of s_h that are joined to s_h by colors in K_F, we have $|V(F \cup F')| \geq n/3 + 2k$. Because r has exactly k many Δ-neighbors, $F \cup F'$ has at least $n/3 + 2k - k = n/3 + k$ many vertices with degree less than Δ. As $d_G(x) \geq \Delta - k + 1$ $\geq 2n/3 + k/2 + 1$, we have $|N_{<\Delta}(x) \cap V(F \cup F')| \geq 2n/3 + k/2 + 1 + n/3 + 2k - n$ $\geq k + 1$, as desired.

By considering the colors on edges joining x and vertices in $N_{<\Delta}(x) \cap V(F \cup F')$, we have the following three cases.

Case 1. There is a vertex $u \in N_{<\Delta}(x) \cap V(F \cup F')$ such that $\tau = \varphi(xu) \in \overline{\varphi}(V(F \cup F'))$.

We first assume that $u \in V(F)$. Now if $\tau \in \overline{\varphi}(V(F))$ and there is a color $\gamma \in \overline{\varphi}(u)$ such that γ and τ are incomparable along $\preceq_F$, we do $1 \to \gamma$ at x. Since u is $(1, \gamma)$-linked to r by Lemma 2.2(a) before the operation, we see that γ is missing at both ends of the edge ux colored by τ after the operation. So u is (γ, τ)-linked to x in the resulting coloring. However, this is a contradiction, as u should be (γ, τ)-linked to the vertex in F with missing color τ by Lemma 2.2(b) and the fact that γ and τ are incomparable along $\preceq_F$. If $\tau \in \overline{\varphi}(V(F))$ and there is a color $\gamma \in \overline{\varphi}(u)$ such that $\tau \prec_F \gamma$ along $\preceq_F$, we similarly do $1 \to \gamma$ at x and get a contradiction with Lemma 2.2(c). In the case $\tau \in \overline{\varphi}(V(F))$ and there is a color $\gamma \in \overline{\varphi}(u)$ such that $\gamma \prec_F \tau$ along $\preceq_F$, we simply do a shifting from rs to $rv_F(\tau)$ and uncolor the edge $rv_F(\tau)$. As a result, there exists a color in $\overline{\varphi}(u)$ that is incomparable with τ and we reach an earlier case.

We now consider the case that $\tau \in \overline{\varphi}(V(F'))$. If $\tau \notin K_F$, then we do $1 \to \tau$ at x. Since r is $(1, \tau)$-linked to $v_{F'}(\tau)$ by Lemma 2.4(b), and the set $\{s \in N(s_h) : \varphi(ss_h) \notin K_F\}$ stays the same, it is easy to see that $F \cup F'$ is still an extended Vizing fan. Similarly by Lemma 2.4(c), s_h is (β, τ)-linked to $v_{F'}(\tau)$. We then do $\tau \to \beta$ at x. Note that F is still a Vizing fan under this new coloring. So by $\beta \to 1$ at x, we reach the earlier case of $\varphi(xu) \in \overline{\varphi}(V(F))$, because s_h is $(1, \beta)$-linked to r. For readers' convenience, in the remainder of this paper we will only give operations performed without repeating each time in details Lemmas 2.2 and 2.4, the set $\{s \in N(s_h) : \varphi(ss_h) \notin K_F\}$, and the resulting extended Vizing fan. Now if $\tau \in K_F$, we do $1 \to \beta \to \tau$ at x. Since $|\overline{\varphi}(s_0)| \geq 2$, by Lemma 2.2(d), there exists a color $\zeta \in \overline{\varphi}(V(F)) \cap K$ with $\zeta \neq \beta$. If $v_{F'}(\tau)$ is obtained through a linear sequence with first vertex v joined to s_h by the color 1, we do $\tau \to \zeta \to 1$ at x following Lemma 2.4(d) and Lemma 2.2(a), where we reach the previous case of

$\varphi(xu) \in \overline{\varphi}(V(F))$. Note that here $\varphi(vs_h)$ might be changed to ζ and the set $\{s \in N(s_h) : \varphi(ss_h) \notin K_F\}$ might change, but u stays in an extended Vizing fan in the new coloring. If $v_{F'}(\tau)$ is obtained through a linear sequence with first vertex v joined to s_h by a color other than 1, we do $\tau \to 1$ at x following Lemma 2.4(d), where we reach the previous case of $\varphi(xu) \in \overline{\varphi}(V(F))$. Here the set $\{s \in N(s_h) : \varphi(ss_h) \notin K_F\}$ might change, but u stays in an extended Vizing fan in the new coloring.

We then assume that $u \in V(F')$. Let γ be a color in $\overline{\varphi}(u)$. If $\gamma \in K_F$, we can just do $1 \to \beta \to \gamma \to \tau$ at x and get a non-elementary extended Vizing fan, a contradiction to Lemma 2.4(a). Therefore, we may assume $\gamma \notin K_F$. If $v_{F'}(\tau)$ and u do not belong to the same linear sequence L added to F' with $u \prec_L v_{F'}(\tau)$, then we have a non-elementary extended Vizing fan by $1 \to \gamma \to \tau$ at x. Thus we may assume $v_{F'}(\tau)$ and u are both belong to a linear sequence L added to F' with $u \prec_L v_{F'}(\tau)$. Let the first vertex of L be v. Since $|\overline{\varphi}(s_0)| \geq 2$, by Lemma 2.2(d), there exists a color $\zeta \in \overline{\varphi}(V(F)) \cap K$ with $\zeta \neq \beta$. We first do $1 \to \beta \to \tau$ at x. Now similarly as before, if $\varphi(vs_h) = 1$, we then do $\tau \to \zeta \to 1$ at x following Lemma 2.4(d) to reach the previous case as $\varphi(xu) = \beta \in \overline{\varphi}(V(F))$. If $\varphi(vs_h) \neq 1$, we then do $\tau \to 1$ at x following Lemma 2.4(d) to reach the previous case as $\varphi(xu) = \beta \in \overline{\varphi}(V(F))$.

Case 2. There is a vertex $u \in N_{<\Delta}(x) \cap V(F \cup F')$ such that $\tau = \varphi(xu) \notin \overline{\varphi}(V(F \cup F'))$ and not all the τ-sequence outside of F is of Type D.

By the assumption of this case, there is a τ-sequence L outside of F which is of Type A, B, or C. Let $\gamma \in \overline{\varphi}(u)$. We first assume that $u \in V(F)$. Now if L is of Type A or C, then by doing $1 \to \gamma \to \tau$ at x, we have a non-elementary Vizing fan contradicting Lemma 2.5. If L is of Type B with $\overline{\varphi}(V(L)) \cap \{1, \gamma\} \neq \emptyset$, then by doing $1 \to \gamma \to \tau$ at x, we still have $\overline{\varphi}(V(L)) \cap \{1, \gamma\} \neq \emptyset$, reaching a contradiction by resulting either a larger Vizing fan or a non-elementary Vizing fan. Now the remaining case is that L is of Type B, and there exists a color $\eta \in \overline{\varphi}(V(F)) \cap \overline{\varphi}(V(L))$ with $\eta \notin \{1, \gamma\}$. If $u \preceq_F v_F(\eta)$ along $\preceq_F$ does not happen, then by doing $1 \to \gamma \to \tau$ at x, we have a non-elementary Vizing fan contradicting Lemma 2.5. If $u \preceq_F v_F(\eta)$ along $\preceq_F$, then we simply do a shifting from rs to $rv_F(\eta)$ and uncolor the edge $rv_F(\eta)$, reaching the earlier case of $u \preceq_F v_F(\eta)$ not happening.

We then assume $u \in V(F')$. We first consider the case that L is not of Type B with $\{1\} = \overline{\varphi}(V(L)) \cap \overline{\varphi}(V(F))$, or L is of Type B with $\{1\} = \overline{\varphi}(V(L)) \cap \overline{\varphi}(V(F))$ and $x \in V(L)$. If $\gamma \notin K_F$, we just do $1 \to \gamma \to \tau$ at x to get a non-elementary extended Vizing fan, a contradiction to Lemma 2.4(a). Therefore, we have $\gamma \in K_F$. If L is not of Type B with $\{\beta\} = \overline{\varphi}(V(L)) \cap \overline{\varphi}(V(F))$, we have a non-elementary extended Vizing fan by $1 \to \beta \to \gamma \to \tau$ at x, a contradiction to Lemma 2.2(a). Thus we may assume L is of Type B with $\{\beta\} = \overline{\varphi}(V(L)) \cap \overline{\varphi}(V(F))$. Suppose that u is added to F' by a linear sequence L' with first vertex v. Again since $|\overline{\varphi}(s_0)| \geq 2$, by Lemma 2.2(d), there exists a color

$\zeta \in \overline{\varphi}(V(F)) \cap K$ with $\zeta \neq \beta$. Now if $\varphi(vs_h) \neq 1$, we do $1 \to \gamma \to \tau$ at x to get a non-elementary extended Vizing fan, and if $\varphi(vs_h) = 1$, we do $1 \to \zeta \to \gamma \to \tau$ at x to get a non-elementary extended Vizing fan, both give contradictions to Lemma 2.2(a).

Thus we can assume L is of Type B with $\{1\} = \overline{\varphi}(V(L)) \cap \overline{\varphi}(V(F))$ and there is a vertex $z \in V(L)$ with $1 \in \overline{\varphi}(z)$ and $z \neq x$. Clearly $\tau \notin K_F$ as L is of Type B. By the definition of extremal linear sequences outside of F, z is the last vertex of L. Let the first vertex of L be w. Note that here z and w could be the same vertex. Clearly F' and L do not share common vertices, as otherwise $z \in V(F')$ and we have a non-elementary extended Vizing fan. Recall that $\gamma \in \overline{\varphi}(u)$. Now we do $1 \to \beta$ at both x and z following Lemma 2.2(a). As a result, $\beta \in \overline{\varphi}(z) \cap \overline{\varphi}(x)$. Note that $\varphi(rw) = \tau$ and $d(w) < \Delta$, so $\tau \notin K_F$. Therefore, s_h and r must be (β, τ)-linked, as otherwise we have a larger Vizing fan by interchanging β and τ along $C_r(\beta, \tau)$. Now if s_h and x are (β, τ)-linked, we can do $\beta \to \tau$ at z and then do $\beta \to 1$ at x, reaching the earlier case that L is not of Type B. We then assume s_h and z are (β, τ)-linked. In this case, we first do $\beta \to \tau$ at x and then do $\beta \to 1$ at z. Now $\varphi(ux) = \beta$, $\tau \in \overline{\varphi}(x)$, and $1 \in \overline{\varphi}(z)$. We then do a shift along L from w to z, and color the edge rz by 1. For reader's convenience, we switch labels for color 1 and τ to meet the notations we used earlier. After the switching of 1 and τ, we now still have $1 \in \overline{\varphi}(r) \cap \overline{\varphi}(x)$, $\gamma \in \overline{\varphi}(u)$, $\varphi(ux) = \beta$, and $F \cup F'$ is still an extended Vizing fan as $\tau \notin K_F$, which returns to Case 1. Finally we may assume that s_h is (β, τ)-linked to neither z nor x. We then do $\beta \to \tau$ at both z and x. As a result, $\tau \in \overline{\varphi}(x) \cap \overline{\varphi}(z)$ and $\varphi(ux) = \beta$. Recall that $\tau \notin K_F$. If r is $(1, \tau)$-linked to z, then by doing $\tau \to 1$ at x, we reach Case 1. If r is $(1, \tau)$-linked to x, then by doing $\tau \to 1$ at z, we reach the earlier case where we shift along L from w to z, and color the edge rz by 1. If r is $(1, \tau)$-linked to neither z nor x, then by doing $\tau \to 1$ at both z and x, we reach Case 1. This finishes Case 2.

Case 3. All the τ-sequence outside of F is of Type D, where $\tau = \varphi(xu) \notin \overline{\varphi}(V(F) \cup V(F'))$ and $u \in N_{<\Delta}(x) \cap V(F \cup F')$.

Since $|N_{<\Delta}(x) \cap V(F \cup F')| \geq k + 1$ by Claim 1 and $|K_F| < k$, there must exist two vertices u and u^* in $N_{<\Delta}(x) \cap V(F \cup F')$ such that $\tau = \varphi(xu) \notin \overline{\varphi}(V(F) \cup V(F'))$ and $\tau^* = \varphi(xu^*) \notin \overline{\varphi}(V(F) \cup V(F'))$ where the τ-sequence L_1 and τ^*-sequence L_2 are both of Type D ending with the same stopping color of F in K_F.

We claim that one of L_1 and L_2 is a sub-sequence of the other one. Otherwise, since L_1 and L_2 are of Type D both ending with the same stopping color, there exists $\theta \in \overline{\varphi}(v_1) \cap \overline{\varphi}(v_2)$ such that $v_1 \in V(L_1)$, $v_2 \in V(L_2)$, and $v_1 \neq v_2$. Because L_1 and L_2 are of Type D, $\theta \notin \overline{\varphi}(V(F))$. Since $\beta \in \overline{\varphi}(s_h)$, at most one of v_1 and v_2 is (β, θ)-linked to s_h. Thus we may assume that v_1 is not (β, θ)-linked to s_h. Note that if $\theta \notin K_F$, r and s_h must be (β, θ)-linked, as otherwise by switching β and θ along $C_r(\beta, \theta, \varphi)$, we would have a larger Vizing fan. Now by $\theta \to \beta$ at s_1, we have reached Case 2 as the sub-sequence of L_1

ending at v_1 is of Type B and this operation will not change the set $\{s \in N(s_h) : \varphi(ss_h) \notin K_F\}$. Thus we have as claimed.

Now by the above claim, we may assume L_2 is a sub-sequence of L_1, $\tau^* \in \overline{\varphi}(z)$ with $z \in V(L_1)$, and $\tau \notin K_F$. We are going to consider the following three cases depending on which one of u and u^* is in $V(F')$.

We first assume that both u and u^* are in $V(F)$. Similarly as earlier, we may assume that there exist $\gamma \in \overline{\varphi}(u)$ and $\gamma^* \in \overline{\varphi}(u^*)$ such that γ and γ^* are incomparable along $\preceq_F$. Otherwise, say $\gamma \preceq_F \gamma^*$, then by shifting from s_0 to $v_F(\gamma^*)$ and uncolor the edge $rv_F(\gamma^*)$, we have as desired. Now we first do $1 \to \gamma \to \tau$ at x following Lemma 2.2(a) and consider a maximal Vizing fan F^* under this new coloring φ. Since now $\tau \in \overline{\varphi}(u)$, and γ and γ^* were incomparable along $\preceq_F$ earlier, we have $u, u^*, z \in V(F^*)$, $\tau \in \overline{\varphi}(u) \cap \overline{\varphi}(x)$, $\tau^* \in \overline{\varphi}(z)$, $\varphi(xu^*) = \tau^*$, and τ^* and γ^* are incomparable along $\preceq_{F^*}$. Following Lemma 2.2(a), we can do $\tau \to 1 \to \gamma^*$ at x. As a result, F^* is still a Vizing fan and now u^* and x are (γ^*, τ^*)-linked, a contradiction to Lemma 2.2(b).

We then assume that $u \in V(F')$ and $u^* \in V(F)$, or both $u, u^* \in V(F')$ but u and u^* do not satisfy $u \preceq_{L^*} u^*$ for a linear sequence L^* in F'. Note that $\{\gamma, \gamma^*\} \cap \{\tau, \tau^*\} = \emptyset$ as $\tau = \varphi(xu) \notin \overline{\varphi}(V(F) \cup V(F'))$ and $\tau^* = \varphi(xu^*) \notin \overline{\varphi}(V(F) \cup V(F'))$. In the case that $\gamma \notin K_F$, we do $1 \to \gamma \to \tau$ at x following Lemma 2.4(b), and in the case that $\gamma \in K_F$, we do $1 \to \beta \to \gamma \to \tau$ at x following Lemma 2.2(b) and Lemma 2.4(c). Note that the set $\{s \in N(s_h) : \varphi(ss_h) \notin K_F\}$ does not change after the above operations and $\tau \in \overline{\varphi}(u)$ now. Thus if γ was in $\overline{\varphi}(V(L_1))$ and γ was changed to 1 or β by the above operations in $\overline{\varphi}(V(L_1))$, we then have a non-elementary extended Vizing fan contradicting Lemma 2.4(a). Otherwise, since L_2 is a sub-sequence of L_1, $\tau^* \in \overline{\varphi}(z)$ with $z \in V(L_1)$ and $u \preceq_{L^*} u^*$ is not satisfied for any L^* in F', the new extended Vizing fan will contain z, u, u^* while $\tau^* \in \overline{\varphi}(z)$, $\tau \in \overline{\varphi}(x)$ and $\varphi(u^*x) = \tau^*$. Now since $\tau \notin K_F$, we just do $\tau \to 1$ at x following Lemma 2.4(b) to reach Case 1 as $\varphi(u^*x) = \tau^* \in \overline{\varphi}(z)$.

Finally we assume that $u^* \in V(F')$ and $u \in V(F)$, or both $u, u^* \in V(F')$ but u and u^* do not satisfy $u^* \preceq_{L^*} u$ for a linear sequence L^* in F'. Note that $\{\gamma, \gamma^*\} \cap \{\tau, \tau^*\} = \emptyset$ as $\tau = \varphi(xu) \notin \overline{\varphi}(V(F) \cup V(F'))$ and $\tau^* = \varphi(xu^*) \notin \overline{\varphi}(V(F) \cup V(F'))$. Similar as before, in the case that $\gamma^* \notin K_F$, we do $1 \to \gamma^* \to \tau^*$ at x following Lemma 2.4(b), and in the case that $\gamma^* \in K_F$, we do $1 \to \beta \to \gamma^* \to \tau^*$ at x following Lemma 2.2(b) and Lemma 2.4(c). Note that the set $\{s \in N(s_h) : \varphi(ss_h) \notin K_F\}$ does not change after the above operations, and $\tau^* \in \overline{\varphi}(u^*)$ and $\tau^* \in \overline{\varphi}(x)$. Moreover, we now have $\tau^* \in \overline{\varphi}(u^*)$, $\tau^* \in \overline{\varphi}(z)$, $\varphi(u^*x) = \gamma^*$, and vertices u, u^* and the sub-sequence of L_1 after z is contained in an extended Vizing fan after the above operations. Thus if γ^* was a missing color in the sub-sequence of L_1 after z and γ^* was changed to 1 or β by the above operations as a missing color in the sub-sequence of L_1, we then have a non-elementary extended Vizing fan contradicting Lemma 2.4(a). If γ^* was a missing color in the sub-sequence of L_1 until z and γ^* was changed to 1 or β by the above operations as a missing color in the sub-sequence of L_1, we then do $\tau^* \to 1$ at x when $\tau^* \notin K_F$ and $\tau^* \to \beta \to 1$ at x when $\tau^* \in K_F$. Now $1 \in \overline{\varphi}(x)$, $\tau^* \in \overline{\varphi}(u)$, $\varphi(xu) = \tau$, u and u^* are in an extended Vizing fan while there is τ-sequence of either Type A or B, reaching Case 1. If the above two possibilities did not happen, then we still have $\tau^* \in \overline{\varphi}(u^*) \cap \overline{\varphi}(z) \cap \overline{\varphi}(x)$, $\tau = \overline{\varphi}(ux)$, u and u^* are still in an extended Vizing fan, and the τ-sequence L_1 stays the same containing the vertex z. Now if $\tau^* \notin K_F$, we do $\tau^* \to 1$ at both x and z following Lemma 2.4(b) to reach Case 1, as now there is a τ-sequence of Type B. In the case that $\tau^* \in K_F$, we do $\tau^* \to \beta \to 1$ at both x and z following Lemma 2.4(c) and Lemma 2.2(b) to reach Case 1, as now there is a τ-sequence of Type B.

□

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